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Generalized Rational Approximation in Interpolating Subspaces of $L_p(\mu)$

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We extend the geometric approach of Cheney and Loeb in [2] to the problem of approximation in $L_p(\mu)$ by "admissable" generalized rational functions. We obtain a characterization for locally best approximations and find the interpolating condition sufficient for their local unicity. Our results are comparable to those for the linear approximation problem as investigated by Singer and Ault, Deutsch, Morris, and Olson.

1. INTRODUCTION

A set X, a σ-algebra Σ of subsets of X, a σ-finite measure μ on Σ, and a topology τ on X are prescribed. Assume further that (X, τ) is a compact Hausdorff space. Denote by $L_p(\mu) \equiv L_p(X, \Sigma, \mu)$ the Banach space of (equivalence classes of) measurable real-valued functions on X for which

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p} < \infty, \qquad 1 \leqslant p < \infty.$$

We denote by $C(\mu)$ the subspace of $L_p(\mu)$ consisting of continuous functions.

Let $f, g_1, \ldots, g_n, h_1, \ldots, h_m \in C(\mu)$. Define R_m^n, a subspace of $C(\mu)$, by

$$R_m^n \equiv \left\{ r(x) := \frac{\sum_{i=1}^n \alpha_i g_i(x)}{\sum_{i=1}^m \beta_i h_i(x)}, \qquad \sum_{i=1}^m \beta_i h_i(x) > 0 \text{ on } X \right\}.$$

In particular cases, cf. [3, 4, 8, 9], we know there exists at least one $r_0 \in R_m^n$ such that

$$\|f - r_0\|_p = \left(\int_X |f(x) - r_0(x)|^p \, d\mu\right)^{1/p} = \inf_{r \in R_m^n} \|f - r\|_p .$$

For these cases, r_0 is a globally best approximation to f from R_m^n. In general, the subset $T \equiv [r' \in R_m^n \mid \|f - r'\|_p = \|f - r_0\|_p]$ is nonconvex and standard convexity arguments for proving uniqueness of a global best approximation, suitably normalized, do not hold true. For further discussion, see [5].

For the same reason, it is conceivable that to a coefficient vector $[a_1, ..., a_n, b_1, ..., b_m]$ there corresponds an approximation in R_m^n which is best only in a local vicinity of the coefficient vector.

Our discussion will center on such locally best approximations whenever they occur, whether or not their existence is assured, and we shall obtain sufficient conditions for their unicity in a local vicinity of the coefficient vector.

Of course, every global best approximation satisfies the properties of a locally best approximation.

Brosowski [10], in generalizing the Kolmogoroff criterion, leaves open the question of a necessary condition to be satisfied by best L_p rational approximations when R_m^n is nondegenerate.[1]

We now attempt to answer this.

2. Basic Definitions and Notations

We introduce the concept of local best L_p approximation with respect to a domain.

We shall constrain competing approximations by introducing in condition (1) below an ϵ^*-sphere in E^m (Euclidean m space), i.e., $\{\mathbf{d} \in E^m \mid \|\mathbf{d}\| \leqslant \epsilon^*\}$ where $\|\cdot\|$ is any l_p norm $1 \leqslant p \leqslant \infty$. Furthermore, we exclude cases of vanishing denominators on X by the strong condition (2).

Let D be a fixed nonempty bounded domain in E^{n+m} which includes the origin.

Let

$$r_0(x) := \frac{\sum_{i=1}^n a_i g_i(x)}{\sum_{i=1}^m b_i h_i(x)} .$$

Definition 2.0. $r_0(x)$ is a locally best L_p approximation to $f \in C(\mu)$ with respect to D, if there exists an $\epsilon^* > 0$ such that for all $(c_1, ..., c_n, d_1, ..., d_m) \in D$ satisfying

$$(1) \quad \|\mathbf{d}\| \leqslant \epsilon^* \qquad \text{and} \qquad (2) \quad \left|\epsilon^* \sum_{i=1}^m d_i h_i(x)\right| < \sum_{i=1}^m b_i h_i(x) \text{ on } X,$$

[1] In [14], Brosowski has obtained some related results which are further developed here.

we have $\|f - r_\lambda\|_p \geqslant \|f - r_0\|_p$ for all λ, $|\lambda| \leqslant \epsilon^*$, where

$$r_\lambda(x): = \frac{\sum_{i=1}^n (a_i + \lambda c_i)\, g_i(x)}{\sum_{i=1}^m (b_i - \lambda d_i)\, h_i(x)}.$$

We shall use the following abbreviations. We set

$$q_m(\lambda, \mathbf{d}, x): = \sum_{i=1}^m (b_i - \lambda d_i)\, h_i(x) \qquad \text{and} \quad q_m(x): = \sum_{i=1}^m b_i h_i(x).$$

We let $P \equiv \text{span}[g_1, ..., g_n]$ and $Q \equiv \text{span}[h_1, ..., h_m]$. We shall adopt the following notation.

Let $L_p^*(\mu)$ denote the strong dual space of $L_p(\mu)$, i.e., the set of bounded linear functionals defined on $L_p(\mu)$ together with the norm

$$\| L \| = \sup_{\|f\|_p \leqslant 1} | Lf | \qquad \text{for} \quad f \in L_p(\mu) \qquad \text{and} \quad L \in L_p^*(\mu).$$

We remark that $L_1^*(\mu)$ is isometrically isomorphic to the space $L_\infty(\mu)$ of essentially bounded measurable functions via the correspondence

$$L(f) = \int_X fg\, d\mu \qquad \text{for} \quad L \in L_1^*(\mu) \qquad \text{and} \quad g \in L_\infty(\mu)$$

for all $f \in L_1(\mu)$.

Let $S \equiv [L \in L_p^*(\mu) \mid \| L \| \leqslant 1]$ be the unit sphere of $L_p^*(\mu)$, ext(S) be the extreme points of S and

$$E_0(S) \equiv [L \in \text{ext}(S) \mid L(f - r_0) = \|f - r_0\|_p].$$

We note that $E_0(S)$ is nonempty by the Hahn–Banach theorem.

Definition. Let X be a normed linear space and X^* its dual space. The weak* topology is the weakest topology on X^* such that all linear functionals generated by X are continuous.

Examples. The linear functional, mapping $L_p^*(\mu) \to E^1$ and defined by

$$\frac{\alpha_i g_i}{q_m}(L) = L\left(\frac{\alpha_i g_i}{q_m}\right) \qquad i = 1, ..., n,$$

is continuous on $L_p^*(\mu)$.

Define likewise the $n + m$ dimensional mapping $\mathbf{f}$: $L_p^*(\mu) \to E^{n+m}$ where

$$\mathbf{f} \equiv \left(\frac{\alpha_1 g_1}{q_m}, ..., \frac{\alpha_n g_n}{q_m}, \frac{\beta_1 r_0 h_1}{q_m}, ..., \frac{\beta_m r_0 h_m}{q_m}\right)^T.$$

Then $\mathbf{f}$ is continuous on $L_p^*(\mu)$.

Remark 2.1. $\mathbf{f}$ maps compact subsets of $L_p{}^*(\mu)$ to compact subsets of E^{n+m}.

Remark 2.2. If we define the continuous mapping $p: E^{n+m} \to E^1$ by $p(\mathbf{w}) = \sum_{i=1}^{n+m} w_i$, the composite map $p \circ \mathbf{f}$ is a continuous real valued function and achieves its minimum on a compact subset of $L_p{}^*(\mu)$.

It is well known that S is compact in the weak* topology.

However, the subsequent development of the theory necessitates the additional assumption that ext(S) be closed. Circumstances under which this is the case will be discussed in Section 5. As a consequence $E_0(S)$ is also compact.

3. Characterization of Locally Best L_p Approximations

The following lemma is fundamental to our argument. A proof may be found in [11, p. 19].

Lemma 3.1. *Let A be a compact set in E^N. For all $a \in A$, the system of inequalities $(a, v) > 0$ is inconsistent if and only if $\mathbf{0} \in$ convex hull $[A]$ where $\mathbf{0}$ denotes the origin of N-space.*

Before characterizing r_0, we observe that the linear subspace spanned by $\{g_1, ..., g_n, r_0h_1, ..., r_0h_m\}$ can have dimension at most $n + m - 1$.

Theorem 3.2. *Let $\phi_1, ..., \phi_N$ be a basis for $P/q_m + r_0(Q/q_m)$.*
Let A be the compact set (by Remark 2.1) in E^N

$$[(L(\phi_1), ..., L(\phi_N))^T \text{ over all } L \in E_0(S)].$$

If $r_0(x)$ is a locally best L_p approximation to $f(x)$ from $R_m{}^n$, then

$$\mathbf{0} \in \text{convex hull } [A].$$

If, on the other hand, convex hull $[A]$ is a body in N-space and

$$\mathbf{0} \in \text{interior convex hull } [A]$$

then $r_0(x)$ is a locally best L_p approximation to $f(x)$ from $R_m{}^n$.

Proof of Sufficiency. Suppose r_0 is not a locally best L_p approximation to f. Then $\forall \epsilon > 0$, $\exists \lambda$, $0 < |\lambda| \leqslant \epsilon$ and $\exists (c_1, ..., c_n, d_1, ..., d_m) \in D$ with $\|\mathbf{d}\| \leqslant \epsilon$ and $|\epsilon \sum_{i=1}^m d_ih_i(x)| < \sum_{i=1}^m b_ih_i(x)$ on X and

$$\left\| f - \frac{\sum_{i=1}^n (a_i + \lambda c_i)\, g_i(x)}{\sum_{i=1}^m (b_i - \lambda d_i)\, h_i(x)} \right\|_p < \left\| f - \frac{\sum_{i=1}^n a_i g_i(x)}{\sum_{i=1}^m b_i h_i(x)} \right\|_p .$$

Now for all $L \in S$ we have $L(f - r_\lambda) \leqslant \|f - r_\lambda\|_p$ and so for $L \in E_0(S)$

$$L(f - r_\lambda) < L(f - r_0).$$

Therefore, by simple manipulation

$$0 < \operatorname{sgn} \lambda \sum_{i=1}^{n} c_i L\left(\frac{g_i}{q_m(\lambda, \mathbf{d})}\right) + \operatorname{sgn} \lambda \sum_{i=1}^{m} d_i L\left(\frac{r_0 h_i}{q_m(\lambda, \mathbf{d})}\right).$$

Therefore,

$$\mathbf{0} \notin \text{convex hull} \left[\left(L\left(\frac{q_m}{q_m(\lambda, \mathbf{d})}\phi_1\right), \ldots, L\left(\frac{q_m}{q_m(\lambda, \mathbf{d})}\phi_N\right)\right)^T \text{ over all } L \in E_0(S)\right]$$

by Lemma 3.1.

But by assumption,

$$\mathbf{0} \in \text{interior convex hull } [A].$$

Hence, by continuity (see Appendix), $\exists \epsilon_1 > 0$ such that $\forall \lambda$, $0 < |\lambda| \leqslant \epsilon_1$ and $\forall \mathbf{d}$, $\|\mathbf{d}\| \leqslant \epsilon_1$ we have $\sum_{i=1}^{m} b_i h_i(x) > |\epsilon_1 \sum_{i=1}^{m} d_i h_i(x)|$ on X and $\mathbf{0} \in$ convex hull of

$$\left[\left(L\left(\frac{q_m}{q_m(\lambda, \mathbf{d})}\phi_1\right), \ldots, L\left(\frac{q_m}{q_m(\lambda, \mathbf{d})}\phi_N\right)\right)^T \text{ over all } L \in E_0(S)\right].$$

Hence, we obtain a contradiction.

Proof of Necessity. Suppose $\mathbf{0}$ does not lie in the convex hull. By Lemma 3.1 $\exists$ scalars $\alpha_1', \ldots, \alpha_n', \beta_1', \ldots, \beta_m'$, and positive constants $\gamma'(L)$ such that for all $L \in E_0(S)$

$$\sum_{i=1}^{n} \alpha_i' L\left(\frac{g_i}{q_m}\right) + \sum_{k=1}^{m} \beta_k' L\left(\frac{r_0 h_k}{q_m}\right) = \gamma'(L) > 0.$$

Divide through by a scaling factor $t > 0$ to be determined later, and rerepresent the new constants and scalars by omitting the prime.

Set

$$\gamma^*(t) = \min_{L \in E_0(S)} \gamma(L) > 0.$$

In view of Remark 2.2, this minimum is achieved.

Let $\boldsymbol{\beta} := [\beta_1, \ldots, \beta_m]$.

For any $\epsilon^* > 0$, choose t so that $\boldsymbol{\beta}$ satisfies conditions (1) and (2) of

Definition 2.0 and $(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m) \in D$. Then we assert $\exists \lambda^*, 0 < \lambda^* \leqslant \epsilon^*$ such that $\forall \lambda \in I \equiv [-\lambda^*, \lambda^*]$

$$\sum_{i=1}^{n} \alpha_i L\left(\frac{g_i}{q_m(\lambda, \boldsymbol{\beta})}\right) + \sum_{k=1}^{m} \beta_k L\left(\frac{r_0 h_k}{q_m(\lambda, \boldsymbol{\beta})}\right) > \frac{\gamma^*(t)}{2} \text{ for all } L \in E_0(S).$$

Set $e(x) := f(x) - r_0(x)$. Let $S^0(t)$ be the set of $L \in \text{ext}(S)$ such that for any $\lambda \in I$,

$$\sum_{i=1}^{n} \alpha_i L\left(\frac{g_i}{q_m(\lambda, \boldsymbol{\beta})}\right) + \sum_{k=1}^{m} \beta_k L\left(\frac{r_0 h_k}{q_m(\lambda, \boldsymbol{\beta})}\right) > \frac{\gamma^*(t)}{2}.$$

The following may be stated about S^0.

(i) S^0 is independent of λ;

(ii) S^0 includes $E_0(S)$ and is, hence, nonempty;

(iii) S^0 is open in the weak* topology.

Let S^c be the complement of S^0 in ext(S). S^c is closed in the weak* topology and, therefore, compact. Hence, $L(f - r_0)$ achieves on S^c a supremum $\kappa < \| e \|_p$. If S^c is empty, set $\kappa = 0$.

For all $\lambda \in I$, the function

$$r_\lambda(x) := \frac{\sum_{i=1}^{n} (a_i + \lambda\alpha_i) g_i(x)}{\sum_{k=1}^{m} (b_k - \lambda\beta_k) h_k(x)}$$

is well defined.

We proceed to show that there exists $\hat{\epsilon} > 0$ such that for all λ, $0 < \lambda \leqslant \hat{\epsilon}$, r_λ is a better approximation to f than r_0.

Observing first that for $0 \leqslant |\lambda| \leqslant \lambda^*$

$$\sup_{x \in X} \left| \frac{1}{q_m(\lambda, \boldsymbol{\beta}, x)} \right| < \frac{1}{\inf_{x \in X} [\sum_{k=1}^{m} b_k h_k(x) - |\lambda^* \sum_{k=1}^{m} \beta_k h_k(x)]}$$
$$< \infty,$$

we assert $\exists \hat{\epsilon}$, $0 < \hat{\epsilon} \leqslant \lambda^*$ such that for all λ $0 \leqslant |\lambda| \leqslant \hat{\epsilon}$ and for any $L \in S$

$$| L(r_0 - r_\lambda)| < \| e \|_p - \kappa.$$

Take any λ satisfying $0 < \lambda \leqslant \hat{\epsilon}$. Consider $L \in S^c$

$$\begin{aligned} L(f - r_\lambda) &= L(f - r_0) + L(r_0 - r_\lambda) \\ &\leqslant \kappa + | L(r_0 - r_\lambda)| \\ &< \| e \|_p . \end{aligned}$$

Consider $L \in S^0$

$$L(f - r_\lambda) = L(f - r_0) + L(r_0 - r_\lambda)$$
$$= Le - \lambda L \left(\frac{\sum_{i=1}^{n} \alpha_i g_i + r_0 \sum_{k=1}^{m} \beta_k h_k}{q_m(\lambda, \boldsymbol{\beta})} \right)$$
$$< \| e \|_p - \frac{\lambda \gamma^*(t)}{2}.$$

Hence, for $0 < \lambda \leqslant \hat{\epsilon}$, $\exists r_\lambda$ such that $L(f - r_\lambda) < \| e \|_p$ for all $L \in \text{ext}(S)$. But there exists at least one $L \in \text{ext}(S)$ satisfying $L(f - r_\lambda) = \| f - r_\lambda \|_p$ (cf. [13, p. 65, Corollary 14]). Hence, $\| f - r_\lambda \|_p < \| f - r_0 \|_p$.

COROLLARY 3.3. *A necessary condition for r_0 to be a locally best L_p approximation to $f(x)$ from $R_m{}^n$ is*

$$\min_{L \in E_0(S)} L(\phi) \leqslant 0$$

for all $\phi \in P/q_m + r_0(Q/q_m)$.

A sufficient condition for r_0 to be a locally best L_p approximation to $f(x)$ from $R_m{}^n$ is that

$$\min_{L \in E_0(S)} L(r_\lambda - r_0) \leqslant 0$$

for all r_λ satisfying the conditions of Definition 2.0. The sufficiency condition has been previously stated by Brosowski [10] for the global case.

4. LOCALLY BEST APPROXIMATIONS IN INTERPOLATING SUBSPACES

So far, our argument has been developed for

$$L_p(X, \Sigma, \mu) \text{ space} \qquad 1 \leqslant p < \infty.$$

However, if we want to strengthen our characterization of a locally best approximation we shall make an additional assumption called the interpolating condition. When this is valid, we may assert unicity in a neighborhood of the corresponding coefficient vector. In fact, strong unicity in the sense of Newman and Shapiro [6] is exhibited.

For the strictly convex spaces $L_p(X, \Sigma, \mu)$, $1 < p < \infty$, we know that no interpolating subspaces exist (cf. [1, Theorem 3.1]).

DEFINITION. An atom is a set $A \in \Sigma$ with $0 < \mu(A) < \infty$ and such that $B \in \Sigma$, $B \subset A$ implies that either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

DEFINITION. Let $M \equiv \text{span}[\phi_1, ..., \phi_N]$ be an N-dimensional subspace of $L_1(\mu)$. M is called an interpolating subspace if for each set of N independent functionals $L_1, ..., L_N$ in ext(S) the following condition holds

$$\det[L_i(\phi_j)] \neq 0.$$

LEMMA 4.1. *The space $L_1(X, \Sigma, \mu)$ contains an interpolating subspace of dimension $n > 1$ if and only if X is the union of at least n atoms (cf.* [1, *Theorem* 3.3]).

THEOREM 4.2. *Let*

$$p_n \in P \equiv \text{span}[g_1, ..., g_n]$$

and

$$q_m \in Q \equiv \text{span}[h_1, ..., h_m]$$

and suppose $r_0 := p_n/q_m \in R_m{}^n$.

Suppose

(a) *r_0 is a locally best L_1 approximation to f with respect to a given domain D and for an $\epsilon^* > 0$ and*

(b) *$P/q_m + r_0(Q/q_m)$ is an N-dimensional interpolating subspace of $L_1(\mu)$.*

Then

(i) *There exist exactly $N + 1$ independent functionals $L_1, ..., L_{N+1}$ in $E_0(S)$.*

(ii) *0 is the only element ϕ of $P/q_m + r_0(Q/q_m)$ having the property $L_i\phi \geqslant 0$, $L_i \in E_0(S)$, $i = 1, ..., N + 1$.*

(iii) *$\exists\sigma$, $0 < \sigma \leqslant \epsilon^*$ such that $\forall\lambda$, $|\lambda| \leqslant \sigma$ and for all* $\mathbf{d}$ *satisfying conditions* (1) *and* (2) *of Definition* 2.0.

$$\frac{P}{q_m(\lambda, \mathbf{d})} + r_0 \frac{Q}{q_m(\lambda, \mathbf{d})}$$

is an interpolating subspace.

(iv) *r_0 is a unique locally best approximation in the set*

$$U(r_0, D, \epsilon^*, \sigma) \equiv \Big\{ r_\lambda \mid (c_1, ..., c_n, d_1, ..., d_m) \in D,$$

$$\|\mathbf{d}\| \leqslant \epsilon^*, \left| \epsilon^* \sum_{i=1}^{m} d_i h_i(x) \right| < \sum_{i=1}^{m} b_i h_i(x) \text{ on } X, |\lambda| \leqslant \sigma \Big\}.$$

(v) *There exists a constant* $\gamma(f) > 0$ *such that*

$$\forall r_\lambda(x) \in U(r_0, D, \epsilon^*, \sigma),$$
$$\| f - r_\lambda \|_1 \geqslant \| f - r_0 \|_1 + \gamma(f) \| r_\lambda - r_0 \|_1 .$$

Proof. (i) Let $\phi_1, ..., \phi_N$ be a basis for $P/q_m + r_0(Q/q_m)$. By Theorem 3.2 the origin of N space lies in the convex hull of the set

$$[(L_i(\phi_1), ..., L_i(\phi_N))^T \mid L_i \in E_0(S), i = 1, ..., k].$$

By Caratheodory's theorem $k \leqslant N + 1$. Now for each j, $0 = \sum_{i=1}^{k} \theta_i L_i(\phi_j)$ with $\theta_i \geqslant 0$. Hence, by the interpolating condition, $k \geqslant N + 1$ and so $k = N + 1$. Furthermore, the origin cannot lie on the boundary, for then k would be equal to N. Hence, the origin of N space lies in the interior of the convex hull of the set

$$[(L_i(\phi_1), ..., L_i(\phi_N))^T \mid L_i \in E_0(S), i = 1, ..., N + 1].$$

(ii) Suppose ϕ is a nonzero element of $P/q_m + r_0(Q/q_m)$

$$\phi = \sum_{j=1}^{N} a_j \phi_j ,$$

$$L_i \phi = \sum_{j=1}^{N} a_j L_i(\phi_j).$$

Now

$$0 = \sum_{i=1}^{N+1} \theta_i L_i(\phi_j)$$

and multiplying this equation by a_j and summing over j

$$0 = \sum_{i=1}^{N+1} \theta_i \sum_{j=1}^{N} a_j [L_i(\phi_j)],$$

$$0 = \sum_{i=1}^{N+1} \theta_i L_i \phi.$$

By the interpolating condition at most $N - 1$ of the numbers $L_i \phi$ can vanish. Hence, at least one of the $L_i \phi$ is positive and at least one is negative.

Hence, ϕ is zero.

(iii) Let λ, and $\mathbf{d}$ be sufficiently small. Then

$$\tilde{\phi}_i(\lambda, \mathbf{d}) = \frac{q_m}{q_m(\lambda, \mathbf{d})} \phi_i \qquad i = 1, ..., N$$

is a basis for

$$\frac{P}{q_m(\lambda, \mathbf{d})} + r_0 \frac{Q}{q_m(\lambda, \mathbf{d})}.$$

By continuity of determinants, we have

$$\frac{P}{q_m(\lambda, \mathbf{d})} + r_0 \frac{Q}{q_m(\lambda, \mathbf{d})}$$

is an interpolating subspace.

(iv) Let $r_\lambda(x) \in U(r_0, D, \epsilon^*, \sigma)$ be another locally best L_1 approximation to f in the vicinity of r_0. Take

$$\phi := r_0 - r_\lambda \in \frac{P}{q_m(\lambda, \mathbf{d})} + r_0 \frac{Q}{q_m(\lambda, \mathbf{d})},$$

then for $L_i \in E_0(S)$,

$$\begin{aligned} L_i(r_0 - r_\lambda) &= L_i(f - r_\lambda) - L_i(f - r_0) \\ &\leqslant 0 \qquad i = 1,..., N+1. \end{aligned}$$

But from (i) (and the Appendix)

$$\mathbf{0} \in \text{convex hull } [(L_i\tilde{\phi}_1(\lambda, \mathbf{d}),..., L_i\tilde{\phi}_N(\lambda, \mathbf{d}))^T \; i = 1,..., N+1].$$

Hence, by (ii) $r_0 \equiv r_\lambda$.

(v) The proposition is trivial for the case $f \in R_m{}^n$. Otherwise, for $0 < |\lambda| \leqslant \sigma$ define for the set $U(r_0, D, \epsilon^*, \sigma)$

$$\gamma(r_\lambda) = \frac{\| f - r_\lambda \|_1 - \| f - r_0 \|_1}{\| r_\lambda - r_0 \|_1}$$

and suppose to the contrary there exists a sequence

$$\{r_{\lambda_k}\} \in U(r_0, D, \epsilon^*, \sigma) \; r_{\lambda_k} \neq r_0$$

and

$$\gamma(r_{\lambda_k}) \to 0.$$

For $L_i \in E_0(S)$ and

$$\phi \in \frac{P}{q_m(\lambda_k, \mathbf{d}_k)} + r_0 \frac{Q}{q_m(\lambda_k, \mathbf{d}_k)}$$

we have by (iii) and (ii) that

$$c_k = \min_{\|\phi\|_1 = 1} \max_{j=1,...,N+1} L_j\phi > 0 \qquad \text{and} \quad c = \min_k c_k > 0,$$

therefore,

$$\gamma(r_{\lambda_k})\| r_{\lambda_k} - r_0 \|_1 = \| f - r_{\lambda_k} \|_1 - \| f - r_0 \|_1 \geqslant \max_{j=1,\dots,N+1} L_j(r_0 - r_{\lambda_k}),$$

therefore,

$$\gamma(r_{\lambda_k}) \geqslant c_k \geqslant c > 0$$

arriving at a contradiction.

COROLLARY 4.3. *Let $P/q_m + r_0(Q/q_m)$ be an N-dimensional interpolating subspace of $L_1(\mu)$. Then r_0 is a locally best L_1 approximation to f from $R_m{}^n$ if and only if*

$$\min_{L \in E_0(S)} L\phi \leqslant 0, \qquad \text{for all} \quad \phi \in P/q_m + r_0(Q/q_m).$$

We now reformulate Theorem 4.2 in terms of the more familiar "alternation" theorem.

THEOREM 4.4. *Suppose $P/q_m + r_0(Q/q_m)$ is an N-dimensional interpolating subspace of $L_1(\mu)$ with basis $\phi_1, \dots, \phi_N$. Let $L_1, \dots, L_{N+1} \in L_1{}^*(\mu)$.*

Define Δ_i by

$$\Delta_i = \begin{vmatrix} L_1(\phi_1) \cdots & \cdots L_{i-1}(\phi_1)\, L_{i+1}(\phi_1) \cdots & \cdots L_{N+1}(\phi_1) \\ \vdots & & \vdots \\ L_1(\phi_N) \cdots & \cdots L_{i-1}(\phi_N)\, L_{i+1}(\phi_N) \cdots & \cdots L_{N+1}(\phi_N) \end{vmatrix}$$

Then r_0 is a unique locally best L_1 approximation to f if and only if

(i) *there exist $N + 1$ linearly independent functionals $L_1, \dots, L_{N+1}$ in $E_0(S)$.*

(ii) *$\Delta_i \Delta_{i+1} < 0$ for $i = 1, \dots, N$.*

Note that by the interpolating condition $\Delta_i \neq 0$, $i = 1, \dots, N + 1$.

Proof. For necessity it remains to prove (ii). Since by the Characterization theorem

$$\mathbf{0} \in \text{interior convex hull } [(L_i\phi_1, \dots, L_i\phi_N)^T \mid L_i \in E_0(S),\ i = 1, \dots, N + 1],$$

there exist positive scalars θ_i, $i = 1, \dots, N + 1$, and

$$\sum_{i=1}^{N} \theta_i L_i \phi_k = -\theta_{N+1} L_{N+1} \phi_k \qquad \text{for} \quad k = 1, \dots, N.$$

Solving for θ_i by Cramer's rule

$$\theta_i = (-1)^{N-i+1} \frac{\Delta_i}{\Delta_{N+1}} \theta_{N+1}$$

from which the result follows. Conversely, the system of equations

$$\sum_{i=1}^{N} x_i L_i \phi_k = -L_{N+1}\phi_k \qquad k = 1,..., N$$

has a unique solution given by

$$x_i = (-1)^{N-i+1} \frac{\Delta_i}{\Delta_{N+1}}$$

and $\{x_i\}$ are positive $i = 1,..., N$. Hence, $\mathbf{0} \in$ interior convex hull

$$[(L_i\phi_1 ,..., L_i\phi_N)^T \mid L_i \in E_0(S),\ i = 1,..., N + 1].$$

THEOREM 4.5 (generalized de la Vallee–Poussin theorem). *Suppose conditions* (a) *and* (b) *of Theorem* 4.2 *hold. For* $r_\lambda \in U(r_0, D, \epsilon^*, \sigma)$ *let* $\{\tilde{\phi}_i(\lambda, \mathbf{d})\}_{i=1}^N$ *be a basis for*

$$\tilde{M} \equiv \frac{P}{q_m(\lambda, \mathbf{d})} + r_0 \frac{Q}{q_m(\lambda, \mathbf{d})},$$

$L_1 ,..., L_{N+1}$ *be independent functionals in* $E_0(S)$, *and*

$$\Delta_i \equiv \Delta_i(\tilde{\phi}_1(\lambda, \mathbf{d}),..., \tilde{\phi}_N(\lambda, \mathbf{d}), L_1 ,..., L_{N+1})$$

be defined as in Theorem 4.3.

If $\Delta_i L_i(f - r_\lambda)\, \Delta_{i+1} L_{i+1}(f - r_\lambda) < 0$, $i = 1,..., N$, *then*

$$\min_i \mid L_i(f - r_\lambda)\mid \leqslant \| f - r_0 \|_1 .$$

Moreover, if this inequality is actually an equality, then $\mid L_i(f - r_\lambda)\mid = \| f - r_0 \|_1$ *for every i.*

Proof. By Theorem 5.1 in [1]

$$d(f, \tilde{M}) = \max \left| \frac{\sum_{i=1}^{N+1} (-1)^i \Delta_i L_i(f)}{\sum_{i=1}^{N+1} (-1)^i \Delta_i} \right|,$$

where the maximum is over all sets of $N + 1$ independent functionals in $E_0(S)$. If $\min_i \mid L_i(f - r_\lambda)\mid > d(f, \tilde{M})$, then we would obtain a contradiction. Hence,

$$\min_i \mid L_i(f - r_\lambda)\mid \leqslant d(f, \tilde{M}) \leqslant \| f - r_0 \|_1 .$$

5. Applications

We may apply our results to the space $L_1(X, \Sigma, \mu)$ where X is the union of at most countably many atoms, say $X = \bigcup_{i\in I} A_i$. Then it can be shown that ext(S) is weak* closed and that each $L \in$ ext(S) has the representation

$$L(f) = \sum_{i\in I} f(A_i)\, \sigma(A_i)\, \mu(A_i)\, f \in L_1,$$

where $|\sigma(A_i)| = 1$ and $f(A_i)$ denotes the constant value of f a.e. on A_i.

The characterization theorem may be rewritten to take account of the special form of the linear functionals, (cf. [1, Theorem 4.3]). Furthermore, the condition of Lemma 4.1 is satisfied.

A particular subcase is the space $l_1 = L_1(X, \Sigma, \mu)$, where $X = \{1, 2, 3, \ldots\}$, Σ is the collection of all subsets of X, and μ is the counting measure $\mu(B) = \text{card}(B)$.

When approximating with ordinary rational functions, the following lemma is of relevance (cf. [11, p. 162]).

Lemma 5.1. *Let p_n and q_m be ordinary polynomials of degree $\leqslant n - 1$ and $m - 1$, respectively, with no common divisor, and let $r_0 := p_n/q_m \in R_m{}^n$. Then $P/q_m + r_0(Q/q_m)$ is of dimension* $\max\{n + \deg(q_m), m + \deg(p_n)\}$.

Appendix

Lemma. *Let*

$$\mathbf{L}\phi := \{L\phi_i\}_{i=1}^N,$$

$$\mathbf{L}\phi' := \{L\phi_i'\}_{i=1}^N$$

be vectors in Euclidean N space, and let $\mathbf{0}$ denote the origin of N space.

If $\mathbf{0} \in$ interior convex hull $[\mathbf{L}\phi \mid L \in E_0(S)]$, then $\exists \epsilon > 0$ such that for ϕ' satisfying $\| \mathbf{L}\phi - \mathbf{L}\phi' \| < \epsilon$ for all $L \in E_0(S)$, we have $\mathbf{0} \in$ convex hull $[\mathbf{L}\phi' \mid L \in E_0(S)]$.

Proof. Step 1. For a set B in E^n, define $B_t \equiv [\mathbf{x} \in E^n \mid d(\mathbf{x}, E) \leqslant t]$. Note B convex $\Rightarrow B_t$ convex. Define the Hausdorff metric between two nonempty compact sets A, B by

$$\Delta(A, B) = \inf\{t \colon A_t \subseteq B, B_t \subseteq A\}.$$

Let $\mathbf{u}_1, \ldots, \mathbf{u}_{2s}$ be vectors in E^s, and let U be the convex hull in E^s of $[\mathbf{u}_1, \ldots, \mathbf{u}_{2s}]$.

If $\| \mathbf{z}_i - \mathbf{u}_i \| < \epsilon$ for $i = 1,..., 2s$, then $\mathbf{z}_1 ,..., \mathbf{z}_{2s} \in U_\epsilon$. Therefore, $Z :=$ convex hull $[\mathbf{z}_1 ,..., \mathbf{z}_{2s}] \subset U_\epsilon$ since the convex hull of a set is the intersection of all convex sets containing the set. Similarly, $U \subset Z_\epsilon$. Therefore, $\Delta(U, Z) < \epsilon$.

Step 2. Let A be a set in E^N. If $\mathbf{0} \in$ interior convex hull $[A]$, then there exists $2N$ vectors $\mathbf{u}_1 ,..., \mathbf{u}_{2N} \in A$ such that $\mathbf{0} \in$ interior (U) where $U :=$ convex hull $[\mathbf{u}_1 ,..., \mathbf{u}_{2N}]$ (cf. [12, p. 116]). But if $\mathbf{0} \in$ interior (U) then $\exists \epsilon > 0$ such that for any compact convex hull Z

$$\Delta(U, Z) < \epsilon \Rightarrow \mathbf{0} \in Z.$$

For suppose $\mathbf{0} \notin Z$. Then $\exists \epsilon > 0$ such that $\mathbf{0} \notin Z_\epsilon$

$$\Rightarrow \mathbf{0} \notin X \qquad \text{for any set} \quad X \subset Z_\epsilon$$
$$\Rightarrow \mathbf{0} \notin U.$$

Step 3. $\mathbf{0} \in$ interior convex hull $[\mathbf{L}\boldsymbol{\phi} \mid L \in E_0(S)]$

$$\Rightarrow \mathbf{0} \in \text{interior convex hull } [\mathbf{L}_i\boldsymbol{\phi} \mid L_i \in E_0(S),\ i = 1,..., d \cdot d \leqslant 2N]$$

Hence, $\exists \epsilon > 0$ such that for $\boldsymbol{\phi}'$ satisfying $\| \mathbf{L}\boldsymbol{\phi}' - \mathbf{L}\boldsymbol{\phi} \| < \epsilon$ for all $L \in E_0(S)$, we nave

$$\mathbf{0} \in \text{convex hull } [\mathbf{L}_i\boldsymbol{\phi}' \mid L_i \in E_0(S),\ i = 1,..., d \cdot d \leqslant 2N].$$

Therefore, $\mathbf{0} \in$ convex hull $[\mathbf{L}\boldsymbol{\phi}' \mid L \in E_0(S)]$.

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